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Quantum Bertrand duopoly of incomplete information

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Abstract

We study Bertrand's duopoly of incomplete information. It is found that the effect of quantum entanglement on the outcome of the game is dramatically changed by the uncertainty of information. In contrast with the case of complete information where the outcome increases with entanglement, when information is incomplete the outcome is maximized at some finite entanglement. As a consequence, information and entanglement are both crucial factors that determine the properties of a quantum oligopoly.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The game theory as a branch of applied mathematics has been successful in modelling many natural and social phenomena. Recently, people realized that when the games are carried out on quantum systems many exciting features arise due to various quantum characteristics such as quantum superposition and entanglement [1–3], which has been observed in experiment [4]. Since the classical game theory has very important application in economics, it is expected that the quantum game may well shed some new light in this field [5].

In economics, many important markets are neither perfectly competitive nor perfectly monopolized. These markets are usually called oligopolistic or imperfectly competitive and can be analysed based on game theory. The two earliest, and also important, oligopolies are proposed by Cournot [6] and Bertrand [7]. In Cournot's duopoly, it is assumed that each of the firms simultaneously choose the output and leave the price to be determined by the demand curve of the market. Alternatively, Bertrand's duopoly assumes that both firms choose their price simultaneously and the output is determined by the demand curve. However, in the context of classical game theory the Nash equilibrium of these models are similar to the equilibrium of the Prisoner's Dilemma—the Nash equilibrium is Pareto dominated by another strategy profile. Li *et al* have shown that such a dilemma-like situation can be removed from Cournot's duopoly of complete information by quantizing the game [8]. Similar results are obtained by Lo *et al* [9] in Bertrand's duopoly of complete information.

As we know, apart from the games with complete information, there is another distinct kind of game, i.e. the games with incomplete information. In real markets, the incompleteness of information seems somewhat inevitable. It is thus interesting to investigate the oligopolistic games of incomplete information. Cournot's duopoly of incomplete information was first quantized by Du *et al* [10] and was generalized by Chen *et al* [11]. The results indicate that due to the uncertainty of information, the profit function's dependence on entanglement game was dramatically changed. In order to describe the incomplete information two parameters were introduced in [11]—the *information asymmetry* and the *total information uncertainty*.

It is worth pointing out that in practical markets firms are usually price-makers as in Bertrand model instead of quantity-makers as in Cournot's duopoly. A natural question is what role will incomplete information play in the quantum Bertrand's duopoly? We are trying to answer this question in this paper. To quantize the model, we apply Li *et al's* 'minimal' quantization rules [8]. We assume the incomplete information also roots in the uncertainty of cost as in [10, 11]. We studied how the *information asymmetry* and the *total information uncertainty* influence the game, respectively. This paper is organized as follows: in the next section, we give the quantum version of Bertrand's duopoly of incomplete information and obtain the profit function. In section 3, we discuss the influence of the information on the game.

2. The quantum Bertrand's duopoly of incomplete information

We first review the elements of Bertrand's duopoly of complete information. As a model for oligopoly market competition, we suppose two players, say player 1 and player 2, are producing some homogeneous product. They simultaneously decide the price p_1 and p_2 for their products, with cost t_1 and t_2. The quantity each player sells is determined by

$$q_i(p_i, p_j) = a - p_i + bp_j \qquad (0 < b < 1), \tag{1}$$

which is the key assumption of this model. The above expression can be well understood through our common experience. The lower i's price and the higher its opponent's price, the more products i can sell. The parameter b is introduced to describe the interaction of the two players.

The profit function for player i is

$$U_i(p_i, p_j, b) = (a - p_i + bp_j)(p_i - t_i), \tag{2}$$

where $i = 1, 2$.

Li *et al* [8] first proposed a quantization scheme for a game with continuous variables and they successfully quantize Cournot's duopoly. Since the models of Cournot and Bertrand have similar mathematical structure, we expect the same quantization should also work in Bertrand's duopoly. Indeed, the quantization of Bertrand's duopoly has been done by Lo *et al* [9]. In their quantization, they make use of two single-mode electromagnetic fields $|\text{vac}\rangle_1$ and $|\text{vac}\rangle_2$. The direct product of the two states $|\text{vac}\rangle_1 \otimes |\text{vac}\rangle_2$ is the initial state. Then this state undergoes a unitary entanglement operation $\widehat{J}(\gamma) = \exp\left\{-\gamma\left(\widehat{a}_1^\dagger\widehat{a}_2^\dagger - \widehat{a}_1\widehat{a}_2\right)\right\}$ where $\widehat{a}_i^\dagger(\widehat{a}_i)$ is the creation (annihilation) operator of player i's electromagnetic field and γ measures the magnitude of entanglement. Then player i executes its move via a unitary operation $\widehat{D_i}(x_i) = \exp\left\{x_i\left(\widehat{a}_i^\dagger -\right.\right.$

$\widehat{a}_i)/\sqrt{2}\}$. Finally, both players' states are measured after a disentanglement operation $\widehat{J}^{\dagger}(\gamma)$. Thus, the final state is $|\psi_f\rangle = \widehat{J}^{\dagger}(\gamma)(\widehat{D_1}(x_1)\otimes\widehat{D_2}(x_2))\widehat{J}(\gamma)|\text{vac}\rangle_1|\text{vac}\rangle_2 = |p_1\rangle_1|p_2\rangle_2$, which is a product of two coherent states $|p_1\rangle_1$ and $|p_2\rangle_2$. It is straightforward to obtain the relation between p_1, p_2 and x_1, x_2:

$$\begin{cases} p_1(x_1,x_2,\gamma) = x_1\cosh\gamma + x_2\sinh\gamma \\ p_2(x_1,x_2,\gamma) = x_2\cosh\gamma + x_1\sinh\gamma. \end{cases} \tag{3}$$

The observables we measure are the eigenvalue of the final coherent state p_1 and p_2, which represent the price set by the two players. We can see from equation (3) that both p_1 and p_2 are determined by x_1, x_2 when $\gamma \neq 0$. This property leads to the correlation between the two players. When $\gamma = 0$, the quantum game goes back to the classical one.

When the information is incomplete, player i, though clearly knowing its own unit cost t_i, does not know exactly what t_j is. Instead, player i only knows that t_j obeys the distribution $f_j(t_j)$. It is straightforward to obtain the expectation profit for player i:

$$\begin{aligned} U_i(x_1,x_2,b) = \int & f_j(t_j)[(x_i\cosh\gamma + x_j\sinh\gamma) - t_i] \\ & \times [a - (x_i\cosh\gamma + x_j\sinh\gamma) + b(x_j\cosh\gamma + x_i\sinh\gamma)]\,\mathrm{d}t_j. \end{aligned} \tag{4}$$

Since i's strategy depends on its information, which only consists of t_i, f_i and f_j, the optimal strategy for i must take the form $x_i^* = x_i^*(t_i, f, f_j)$. To obtain x_i^* we solve the variation equation, which yields the Bayes–Nash equilibrium

$$\delta_{x_i} U_i(x_1,x_2,b) = 0. \tag{5}$$

The solution is easy to work out[3]:

$$x_i^* = \frac{t_i}{2\cosh\gamma} + \frac{\mathrm{e}^{-\gamma}\cosh\gamma}{2\cosh\gamma - b\,\mathrm{e}^{\gamma}}a - \frac{(\sinh 2\gamma - b\cosh 2\gamma)\,\mathrm{e}^{-\gamma}}{2\cosh\gamma(2\cosh\gamma - b\,\mathrm{e}^{\gamma})}\bar{t}, \tag{6}$$

where we have assumed that $\bar{t}_1 = \bar{t}_2 = \bar{t}$ for simplicity.

Strictly speaking, equation (5) is only a necessary condition for Bayes–Nash equilibrium. However in the case we concern here, it is also sufficient. The second order derivative of $U_i(x_1,x_2,b)$ with respect to x_i is constantly negative. So x_i^* can guarantee $U_i(x_1,x_2,b)$ to be maximum. Thus x_i^* is indeed the Bayes–Nash equilibrium in our problem.

To avoid the complexity caused by specific choice of player i's cost t_i, we shall average over t_i thus we have the average optimal profit:

$$\overline{U_i} = \frac{\cosh\gamma - b\sinh\gamma}{4\cosh\gamma}\overline{\Delta t_i^2} - \frac{\sinh\gamma(\sinh\gamma - b\cosh\gamma)}{4\cosh^2\gamma}\overline{\Delta t_j^2} + \overline{U_i^C}, \tag{7}$$

where $\overline{\Delta t_i^2} = \int f_i(t_i)(t_i - \bar{t})^2\,\mathrm{d}t_i$ is the information fluctuation of player i and $\overline{U_i^C} = (a - \bar{t} + b\bar{t})^2\frac{(\mathrm{e}^{2\gamma}+1)[(1-b)\,\mathrm{e}^{2\gamma}+1+b]}{4[(1-b)\,\mathrm{e}^{2\gamma}+1]^2}$, the superscript C denotes the game of complete information. Note that the game of incomplete information goes back to the complete information case when $\overline{\Delta t_1^2} = \overline{\Delta t_2^2} = 0$.

We can have an alternate representation of incomplete information by defining

$$M_i = \overline{\Delta t_i^2} - \overline{\Delta t_j^2}, \qquad N = \overline{\Delta t_i^2} + \overline{\Delta t_j^2}, \tag{8}$$

where M_i can be regarded as a measure of the *information asymmetry* while N represents the *total information uncertainty*.

[3] From equation (5) we have $x_i^* = \frac{a}{2(\cosh\gamma - b\sinh\gamma)} + \frac{t_i}{2\cosh\gamma} - \frac{\sinh 2\gamma - b\cosh 2\gamma}{2\cosh\gamma(\cosh\gamma - b\sinh\gamma)}\overline{x_j^*}$, averaging over t_i we have two equations about $\overline{x_1^*}$ and $\overline{x_2^*}$. Thus, we have $\overline{x_i^*} = \frac{a\cosh\gamma + \bar{t}(\cosh\gamma - b\sinh\gamma)}{\mathrm{e}^{2\gamma}(1-b)+1}$, substituting into the very first equation can obtain x_i^*

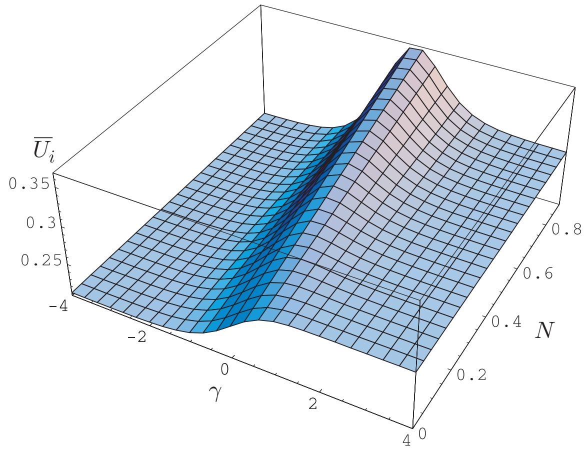

Figure 1. The profit versus quantum entanglement γ and the *total information uncertainty N*. We have taken $a = 1$, $b = 0.5$, $\bar{t} = 0.5$ and $M_i = 0$.

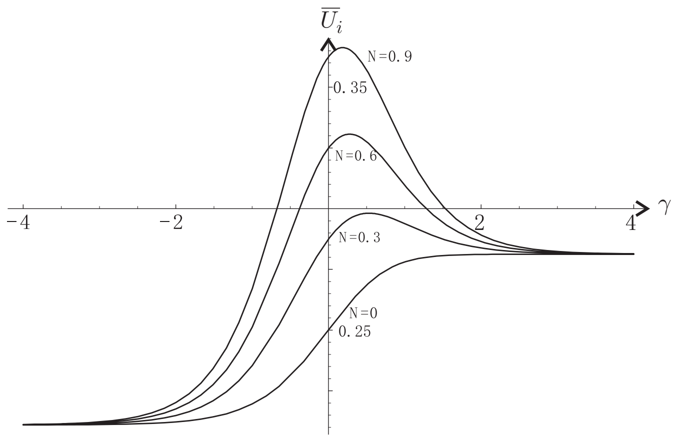

Figure 2. Several cross sections of figure 1 as profit versus quantum entanglement γ with different settings of N.

Then we also have

$$\overline{U_i} = \frac{(\cosh 2\gamma - b \sinh 2\gamma)}{8\cosh^2\gamma} M_i + \frac{1}{8\cosh^2\gamma} N + \overline{U_i^C}. \qquad (9)$$

3. Discussions based on *N* and M_i

The information fluctuations, which are determined by two independent parameters, can be described in several ways, for instance $\overline{\Delta t_1^2}$ and $\overline{\Delta t_2^2}$ or N and M_i. However, in practice, we are particularly interested in the effects brought about by the *total information uncertainty* and *information asymmetry*, which describe the relations of the two players with respect to information. Therefore, the discussions based on N and M_i are presented in this section, respectively.

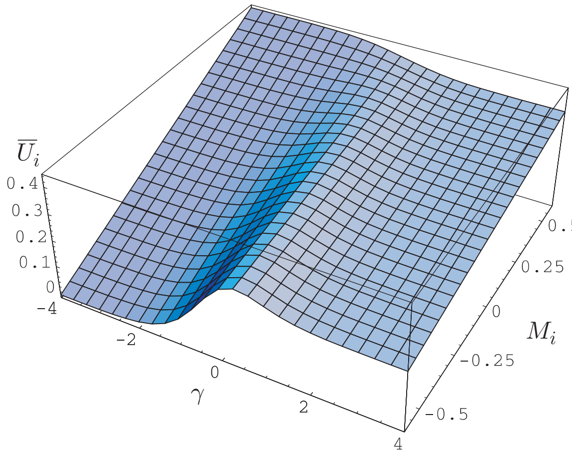

Figure 3. The profit versus quantum entanglement γ and the *information asymmetry* M_i. We have taken $a = 1$, $b = 0.5$, $\bar{t} = 0.5$ and $N = 0.6$.

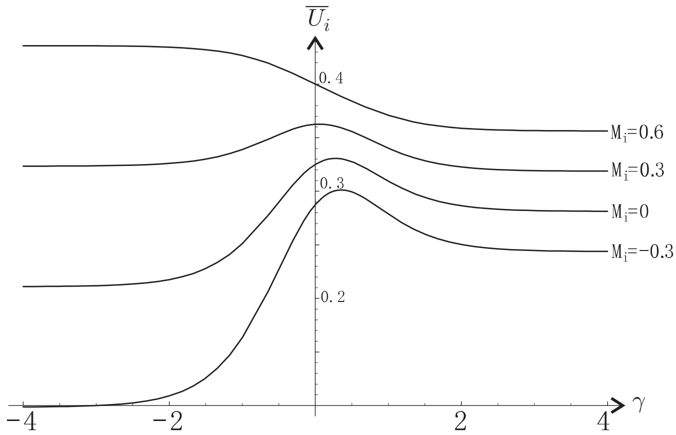

Figure 4. Several cross sections of figure 3 as profit versus quantum entanglement γ with different settings of M_i.

First, we shall study the influence of the *total information uncertainty* N. When information is symmetrically distributed between the two players, i.e. $M_i = 0$, while $N \neq 0$, the effect of incomplete information is purely attributed to the *total information uncertainty* N. Proceeding from equation (9) we plot the profit with respect to N and γ in figure 1, where we take $a = 1$, $b = 0.5$, $\bar{t} = 0.5$ with $M_i = 0$. For the sake of clarity, several cross sections of figure 1 are plotted in figure 2.

From figure 1 as well as figure 2, we can see that when information is complete, i.e. $N = 0$, the profit increases monotonously with γ, which is also obtained in [9]. However, when N comes into play the profit curve exhibits some intriguing features. While the profit in the game of complete information reaches its maximum when $\gamma \to \infty$, in the incomplete information case the profit reaches its peak at some finite $\gamma_{\max}$. Seen from figures 1 and 2, as N increases $\gamma_{\max}$ gradually moves left until it reaches the origin.

In order to investigate the effects of *information asymmetry* M_i, we will fix N and study how the profit function varies with different M_i and the changes of the profit can be purely attributed to the *information asymmetry* M_i. Proceeding from equation (9), we plot the profit with respect to M_i and γ in figure 3, where we take $a = 1$, $b = 0.5$, $\bar{t} = 0.5$ with $N = 0.6$. Also for clarity, several cross sections of figure 3 are plotted in figure 4.

From figures 3 and 4, we can see that M_i significantly affects the profit in several ways. First of all, a larger M_i globally raises the profit, that is at any γ the larger the M_i the better the profit, which can also be seen directly from equation (9). Secondly, there is a threshold $m = \frac{b(a-\bar{t}+b\bar{t})^2}{2(1-b)}$ that when $M_i < m, \overline{U_i}_{\gamma\to\infty}^{\mathrm{opt}} > \overline{U_i}_{\gamma\to-\infty}^{\mathrm{opt}}$; when $M_i > m, \overline{U_i}_{\gamma\to\infty}^{\mathrm{opt}<\overline{U_i}_{\gamma\to-\infty}^{\mathrm{opt}}}$. Besides, the profit reaches its maximum at some finite $\gamma_{\max}$ and as M_i increases $\gamma_{\max}$ moves left. When $M_i = -N$ profit is maximized at $\gamma = -\infty$. While it is claimed in [9] that the negative entanglement diminishes the profit under the classical limit, in the incomplete information case such a claim is valid only when $M_i < \frac{N}{2b+1} + 2(a - \bar{t} + b\bar{t})^2 \frac{4-(1+b)(2-b)^2}{(2b+1)(2-b)^2}$ as can be obtained from equation (9). Otherwise, negative entanglement will actually improve the profit.

There is another interesting effect coming from M_i, when

$$M_i < -\frac{N + 8\cosh^2\gamma\overline{U_i^C}}{(\cosh 2\gamma - b\sinh 2\gamma)}, \tag{10}$$

$\overline{U_i} < 0$. Since a negative M_i means player i's information is inferior to that of its opponent, the above statement can be interpreted in another way; when a player's information inferiority reaches some extent its profit would be negative and have to retreat from the competition. It is worth noting that such a case only happens in quantum game, because if $\gamma = 0$ equation (10) becomes $M_i < -N - 8\overline{U_i^C}$ but on the other hand from equation (8) we know that $-N < M_i < N$. Therefore, such a case resulted from the combination of entanglement and incomplete information.

According to our observation, when information is incomplete maximum profit is achieved at some certain γ and this certain γ depends on the distribution of information. To our expectation, the entanglement is the crucial factor that makes the quantum model of incomplete information different from the classical one.

4. Conclusion

We have studied the quantum Bertrand duopoly of incomplete information. In particular, we have investigated how various properties of the profit function are affected by the incomplete information, which is represented by N and M_i. It is found that the profit is no longer an increasing function of quantum entanglement, the maximal profit is achieved at some finite $\gamma_{\max}$ determined by N and M_i. We have also found that negative entanglement does not necessarily diminish the profit. Besides, when a player's information is inferior to its opponent to some extent, its profit may become negative. Moreover, considering our previous work on Cournot's duopoly, where information also plays a significant role, we can conclude that information and entanglement are both crucial factors that determine the properties of a quantum oligopoly.

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